Circle Diffeomorphisms: Quasi-reducibility and Commuting Diffeomorphisms

Mostapha Benhenda

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Abstract

In this article, we show two related results on circle diffeomorphisms. The first result is on quasi-reducibility: for a Baire-dense set of α, for any diffeomorphism f of rotation number α, it is possible to accumulate R_α with a sequence $h_n f h_n^{-1}$, h_n being a diffeomorphism. The second result is: for a Baire-dense set of α, given two commuting diffeomorphisms f and g, such that f has α for rotation number, it is possible to approach each of them by commuting diffeomorphisms f_n and g_n that are differentiably conjugated to rotations.

In particular, it implies that if α is in this Baire-dense set, and if β is an irrational number such that (α,β) are not simultaneously Diophantine, then the set of commuting diffeomorphisms (f,g) with singular conjugacy, and with rotation numbers (α,β) respectively, is C^∞-dense in the set of commuting diffeomorphisms with rotation numbers (α,β).

1 Introduction

It is well-known that there are circle diffeomorphisms with Liouville rotation numbers (i.e. non-Diophantine) that are not smoothly conjugated to rotations [1, 7, 8, 9]. A natural question arises, namely, the problem of smooth quasi-reducibility: *given a smooth diffeomorphism f of rotation number α, is it possible to accumulate R_α in the C^∞-norm, with a sequence $h_n^{-1} f h_n$, h_n being a smooth diffeomorphism?* In this case, we say that f is smoothly *quasi-reducible* to R_α. Quasi-reducibility is a question that has been studied by Herman [7, pp.93-99], who showed that for any C^2-diffeomorphism f of irrational rotation number α, it is possible to accumulate R_α in the C^{1+bv}-norm, with a sequence $h_n^{-1} f h_n$, h_n being a C^2-diffeomorphism (i.e. $h_n^{-1} f h_n \to R_\alpha$ in the C^1-norm, and the total variation of $D(h_n^{-1} f h_n - R_\alpha)$ converges towards zero). Quasi-reducibility is also related to a problem solved by Yoccoz [10], who showed that it is possible to accumulate a smooth diffeomorphism f in the C^∞-norm with a sequence $h_n R_\alpha h_n^{-1}$, h_n being a smooth diffeomorphism. However, these two problems are not the same, and the method used by Yoccoz does not directly yield our result. In our case, we determine a Baire-dense set of rotation numbers α such that for any smooth diffeomorphism f of rotation number α, f is smoothly quasi-reducible.

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Connected to the problem of quasi-reducibility is the following question, raised by Mather: *given two commuting C^∞-diffeomorphisms f and g, is it possible to approach each of them in the C^∞-norm by commuting smooth diffeomorphisms that are smoothly conjugated to rotations?* In this paper, we determine a Baire-dense set of rotation numbers α such that if f and g are commuting C^∞-diffeomorphisms, with f of rotation number α, then f and g are accumulated in the C^∞ norm by commuting C^∞-diffeomorphisms that are C^∞-conjugated to a rotation. This result is related to a theorem of Fayad and Khanin [6]. They showed that if (α, α') are simultaneously Diophantine (i.e. there is $C_d > 0$, $\beta \geq 0$ such that for any $p, p' \in \mathbb{Z}$, any $q \geq 1$, $\max(|\alpha - p/q|, |\alpha' - p'/q|) \geq C_d/q^{2+\beta}$. This set includes some pairs (α, α') with α and α' Liouvillean), and if f and g are commuting C^∞-diffeomorphisms, with f and g of rotation numbers α and α' respectively, then f and g are smoothly linearizable. Fayad and Khanin's result implies our result of quasi-reducibility in the particular case when the rotation numbers of f and g are simultaneously Diophantine. However, in general, our result is not implied by theirs. Indeed, our result holds for a set (α, α') that is Baire-dense in $\mathbb{R}^2$ (because α belongs to a Baire-dense set of $\mathbb{R}$ and α' is arbitrary), whereas the set of simultaneously Diophantine numbers is not Baire-dense.[1]

Moreover, for Diophantine rotation numbers, which are of full Lebesgue measure, the question of quasi-reducibility and Mather's problem are trivial, because in this case, the diffeomorphism f is smoothly conjugated to a rotation. Therefore, these two questions remain open for a meagre set of rotation numbers of zero Lebesgue measure.

In order to derive our results, we use estimates of the conjugacy to rotations of diffeomorphisms having rotation numbers of Diophantine constant type. These estimates were obtained in [2].

The circle is denoted $\mathbb{T}^1$. For $r \in \mathbb{R}_+ \cup \{+\infty\}$, we work in the universal cover $D^r(\mathbb{T}^1)$, which is the group of diffeomorphisms f of class C^r of the real line such that $f - Id$ is $\mathbb{Z}$-periodic. For $\alpha \in \mathbb{R}$, we denote $R_\alpha \in D^\infty(\mathbb{T}^1)$ the map $x \mapsto x + \alpha$.

Let $f \in D^0(\mathbb{T}^1)$ be a homeomorphism and $x \in \mathbb{R}$. The sequence $((f^n(x) - x)/n)_{n\geq 1}$ admits a limit independent of x, denoted $\rho(f)$. This limit is called the *rotation number* of f. This is a real number invariant by conjugacy.

Theorem 1.1. *There is a Baire-dense set $A_1 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_1$, there is a sequence $h_n \in D^\infty(\mathbb{T}^1)$ such that $h_n^{-1} f h_n \to R_\alpha$ in the C^∞-topology.*

Theorem 1.2. *There is a Baire-dense set $A_2 \subset \mathbb{R}$ such that for any $f \in D^\infty(\mathbb{T}^1)$ of rotation number $\alpha \in A_2$ and any g of class C^∞ with $fg = gf$, f and g are accumulated in the C^∞-topology by commuting C^∞-diffeomorphisms that are C^∞-conjugated to rotations.*

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[1] The complementary in $\mathbb{R}^2$ of simultaneously Diophantine numbers (noted SD^c) is Baire-dense. Indeed, we have:

$$SD^c = \cap_{k\in\mathbb{N}^*} \cap_{n\in\mathbb{N}^*} \cup_{q\geq n}(A_{q,k} \times A_{q,k})$$

with:

$$A_{q,k} = \left\{\alpha \in \mathbb{R} \,/\text{there is an integer} p \in \mathbb{Z}, \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^k}\right\}.$$

$A_{q,k}$ is open (and so is $A_{q,k} \times A_{q,k}$), and for any integer n, $\cup_{q\geq n}(A_{q,k} \times A_{q,k})$ is dense, because it contains all pairs of rational numbers (if $\alpha = p_1/q_1$ and $\alpha' = p_2/q_2$, then $(\alpha, \alpha') \in (A_{jq_1q_2,k} \times A_{jq_1q_2,k})$ for any $j, k \in \mathbb{N}^*$). Therefore, SD^c is Baire-dense.

Remark 1.3. The proof of theorem 1.1 also gives that $h_n R_\alpha h_n^{-1} \to f$ in the C^∞-topology if $\alpha \in A_1$.

Remark 1.4. Combined with [6, p. 965], theorem 1.2 implies that if $\alpha \in A_2$, and (α,β) are not simultaneously Diophantine, then $S_{\alpha,\beta}$, the set of couples (f,g) of smooth commuting circle diffeomorphisms with singular conjugacies to R_α and R_β respectively, is C^∞-dense in $F_{\alpha,\beta}$, the set of couples (f,g) of smooth commuting circle diffeomorphisms with rotation numbers α and β respectively.

Indeed, our result shows that $O_{\alpha,\beta}$, the set of couples (f,g) of smooth commuting circle diffeomorphisms with smooth conjugacies to R_α and R_β respectively, is C^∞-dense in $F_{\alpha,\beta}$. Moreover, in [6, p.965], for (α,β) not simultaneously Diophantine, Fayad and Khanin described the construction of a couple (f,g) of smooth commuting circle diffeomorphisms with singular conjugacies to R_α and R_β respectively. This construction relies on the method of successive conjugacies, which can be made C^∞-dense in $O_{\alpha,\beta}$ [5].

Moreover, by slightly modifying [7, p.160, p.167], this implies that $(O^1_{\alpha,\beta})^c$, the set of couples (f,g) of smooth commuting circle diffeomorphisms with non-C^1 conjugacies to rotations R_α and R_β, is C^∞-generic in $F_{\alpha,\beta}$. See appendix A for a short proof.

2 Preliminaries

2.1 Basic properties

When the rotation number α of f is irrational, and if f is of class C^2, Denjoy showed that f is topologically conjugated to R_α. However, this conjugacy is not always differentiable. It depends on the Diophantine properties of the rotation number α.

Let $\alpha = a_0 + 1/(a_1 + 1/(a_2 + ...))$ be the development of $\alpha \in \mathbb{R}$ in continued fraction (see [4]). It is denoted $\alpha = [a_0, a_1, a_2, ...]$. Let $p_{-2} = q_{-1} = 0$, $p_{-1} = q_{-2} = 1$. For $n \geq 0$, we define integers p_n and q_n by:

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2}.$$

We have $q_0 = 1$, $q_n \geq 1$ for $n \geq 1$. The rationals p_n/q_n are called the convergents of α. Remember that $q_{n+2} \geq 2q_n$, for $n \geq -1$.

For any real number $\beta \geq 0$, $\alpha \in \mathbb{R} - \mathbb{Q}$ is Diophantine of order β and constant C_d (a set denoted $DC(C_d,\beta)$) if there is a constant $C_d > 0$ such that for any $p/q \in \mathbb{Q}$, we have:

$$\left|\alpha - \frac{p}{q}\right| > \frac{C_d}{q^{2+\beta}}.$$

Each of the following relations characterizes $DC(C_d,\beta)$ (see e.g. [11, pp.50-51]):

1. $|\alpha - p_n/q_n| > C_d/q_n^{2+\beta}$ for any $n \geq 0$
2. $a_{n+1} < \frac{1}{C_d} q_n^\beta$ for any $n \geq 0$
3. $q_{n+1} < \frac{1}{C_d} q_n^{1+\beta}$ for any $n \geq 0$

4. $\alpha_{n+1} > C_d \alpha_n^{1+\beta}$ for any $n \geq 0$.

$DC(C_d, 0)$ is the set of irrational numbers of *constant type* C_d. The first derivative of $f \in D^1(\mathbb{T}^1)$ is denoted Df.

2.2 Some useful lemmas

For any n integer, let $\alpha_n = [a_0, ..., a_n, 1, ...]$.

Let $V_\alpha : \mathbb{N} \to \mathbb{R}$ defined by: $V_\alpha(n) = \max_{0 \leq i \leq n} a_i$. Observe that $\alpha_n \in DC(1/V_\alpha(n), 0)$. We will need the lemma:

Lemma 2.1. *Let α be an irrational number, q_n its convergents and $\alpha_n = [a_0, ..., a_n, 1, ...]$. We have:*

$$|\alpha_n - \alpha| \leq \frac{2}{q_n^2} \leq \frac{4}{2^n}.$$

Proof. Let $\tilde{\alpha}_n = [a_0, ..., a_n, 0, ...]$. By induction, we can show that $\tilde{\alpha}_n = p_n/q_n$. Moreover, $\tilde{\alpha}_n$ is also the n^{th} convergent of α_n. Therefore, by the best rational approximation theorem, $|\alpha - p_n/q_n| \leq 1/q_n^2$ and $|\alpha_n - p_n/q_n| \leq 1/q_n^2$. Moreover, since $q_{n+2} \geq q_n$, then $q_n \geq (\sqrt{2})^{n-1}$.

□

We need the lemma:

Lemma 2.2. *Let $\phi : \mathbb{N} \to \mathbb{R}_+$ be such that $\phi(n) \to_{n \to +\infty} +\infty$. Let*

$$A = \{\alpha \in \mathbb{R} \,/ V_\alpha(n) < \phi(n) \textit{for an infinity of} n\}.$$

Then A is Baire-dense.

Proof. First, we show that for any positive integers n and i,

$A_{i,n} = \{\alpha \text{ such that } a_i < \phi(n)\}$ is open. Let $u(x) = \lfloor x \rfloor$, $v(x) = \frac{1}{x}$ and $w(x) = v(x) - u(v(x))$. We have: $a_{k+1} = v(w^k(x)) - w^{k+1}(x)$. Since v is continuous and u is upper semi-continuous and non-negative, then w is lower semi-continuous. Moreover, w is non-negative. Therefore, w^k and w^{k+1} are also lower semi-continuous and non-negative. Since v is decreasing, then $v \circ w^k - w^{k+1}$ is upper semi-continuous. We conclude that $A_{i,n}$ is open.

Moreover, for any $p \geq 0$,

$$\cup_{n \geq p} \cap_{i \leq n} A_{i,n}$$

is dense. Indeed, since $\phi(n) \to +\infty$, then it contains all numbers of constant type, which are dense. This set is also open and therefore,

$$A = \cap_{p \geq 0} \cup_{n \geq p} \cap_{i \leq n} A_{i,n}$$

is Baire-dense.

□

2.3 Notations

- For any real numbers a and b, $a \vee b$ denotes $\max(a, b)$.
- For ϕ a real $\mathbb{Z}$-periodic C^r function, $0 \leq r < +\infty$, we define:

$$\|\phi\|_r = \max_{0 \leq j \leq r} \max_{x \in \mathbb{R}} |D^j \phi(x)|.$$

  Note that for $f, g \in D^r(\mathbb{T}^1)$, $f - g$ is $\mathbb{Z}$-periodic, and for $1 \leq j \leq r$, $D^j f$ is $\mathbb{Z}$-periodic. For $f \in D^r(\mathbb{T}^1)$, we also define:

$$\|f\|_r = \max\left(\|f - id\|_0, \max_{1 \leq j \leq r} \|D^j f\|_0\right).$$

  Note that the notation $\|f\|_r$ is not a norm when $f \in D^r(\mathbb{T}^1)$, since $D^r(\mathbb{T}^1)$ is not a vector space.

- In all the paper, C denotes a constant depending on u. $W(f)$ denotes the total variation of $\log Df$, and Sf denotes the Schwartzian derivative of f.

2.4 Estimates of the conjugacy

The following theorem gives an estimate of the linearization of a diffeomorphism having a rotation numbers of Diophantine constant type. This estimate, obtained in [2], is necessary to derive our results.

Theorem 2.3. *Let $l \geq 3$ be an integer and $\eta > 0$. Let $f \in D^l(\mathbb{T}^1)$ be of rotation number α, such that α is of constant type C_d. There exists a diffeomorphism $h \in D^{l-1-\eta}(\mathbb{T}^1)$ conjugating f to R_α, and a function B of $C_d, l, \eta, W(f), \|Sf\|_{l-3}$, which satisfy the estimate:*

$$\max\left(\frac{1}{\min Dh}, \|h\|_{l-1-\eta}\right) \leq B(C_d, l, \eta, W(f), \|Sf\|_{l-3}). \tag{1}$$

In particular, we remark that if f_n is a sequence of diffeomorphisms of rotation number α_n, if the sequences $W(f_n)$ and $\|Sf_n\|_{l-3}$ are bounded (this will hold in our case, because we will take $f_n = \lambda_n + f$ for a properly chosen $\lambda_n \in \mathbb{R}$), if $V_\alpha(n) \to +\infty$ and if h_n is the conjugacy to a rotation associated with f_n, then there is a real function $E(V_\alpha(n))$ such that, for n sufficiently large, we have:

$$\max\left(\frac{1}{\min Dh_n}, \|h_n\|_{l-1-\eta}\right) \leq E(V_\alpha(n)).$$

3 Quasi-Reducibility

Theorem 3.1. *Let $l \geq 3$ be an integer, $f \in D^l(\mathbb{T}^1)$ be of rotation number $\alpha \in \mathbb{T}^1$. Let $\eta > 0$ be a real number. There exists a numerical sequence $F(n)$, going to $+\infty$ as $n \to +\infty$, such that, if*

$$\liminf \frac{V_\alpha(n)}{F(n)} = 0$$

then there is a sequence h_n of class $C^{l-1-\eta}$ such that $h_n^{-1} f h_n \to R_\alpha$ in the $C^{l-2-\eta}$-topology.

By applying lemma 2.2, we obtain the corollary:

Corollary 3.2. *There is a Baire-dense set $A_1 \subset \mathbb{R}$ such that if $l \geq 3$ is an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in A_1$ and $\eta > 0$, then f is $C^{l-2-\eta}$-quasi-reducible: there is a sequence $h_n \in D^{l-1-\eta}(\mathbb{T}^1)$ such that $h_n^{-1} f h_n \to R_\alpha$ in the $C^{l-2-\eta}$-topology.*

The idea of the proof of theorem 3.1 is the following. We observe that for any sequence $\phi(n) \to +\infty$, the set of numbers α such that for an infinity of n,

$\sup_{k \leq n} a_k \leq \phi(n)$, is Baire-dense (lemma 2.2).

The truncated sequence of constant type numbers $\alpha_n = [a_0, ..., a_n, 1, ...]$ converges towards α at a controlled speed: $|\alpha - \alpha_n| \leq 4/2^n$ (lemma 2.1).

Following an idea of Herman [7], we perturbate f to $R_{\lambda_n} f = f + \lambda_n$ of rotation number α_n, which is linearizable by a conjugacy h_n (lemma 3.3). By writing:

$$h_n^{-1} f h_n - R_\alpha = h_n^{-1} f h_n - h_n^{-1} R_{\lambda_n} f h_n + R_{\alpha_n} - R_\alpha$$

and by applying the Faa-di-Bruno formula, we obtain a control of the norm of $h_n^{-1} f h_n - R_\alpha$ in function of the norm of h_n, and in function of $|\alpha - \alpha_n|$ (lemma 3.4). Moreover, we have an estimate of the norm of h_n in function of $\sup_{k \leq n} a_k$.

Thus, if we choose the speed of growth of the sequence $\sup_{k \leq n} a_k$ sufficiently small with respect to the speed of convergence of α_n towards α, then $h_n^{-1} f h_n$ converges towards R_α, and f is quasi-reducible.

Proof of theorem 1.1. We let $\eta = l/3$ in corollary 3.2. Since f is smooth, then there is a sequence $(h_{n,l})_{n \geq 0} \in D^\infty(\mathbb{T}^1)$ such that, for any integer $l \geq 3$ fixed,

$$||h_{n,l}^{-1} f h_{n,l} - R_\alpha||_{2(\frac{l}{3}-1)} \to_{n \to +\infty} 0.$$

In particular, there is $n(l)$ such that:

$$||h_{n(l),l}^{-1} f h_{n(l),l} - R_\alpha||_{2(\frac{l}{3}-1)} \leq \frac{1}{l}.$$

Let $h_l = h_{n(l),l}$. Let $\epsilon > 0$, and let $k > 0$ be an integer. There is $l_0 \geq 0$ such that for any $l \geq l_0$, we have: $\epsilon \geq 1/l$, $k \leq 2\left(\frac{l}{3} - 1\right)$ and:

$$||h_l^{-1} f h_l - R_\alpha||_k \leq ||h_l^{-1} f h_l - R_\alpha||_{2(\frac{l}{3}-1)} \leq \frac{1}{l} \leq \epsilon.$$

Therefore, $h_l^{-1} f h_l \to_{l \to +\infty} R_\alpha$ in the C^k-topology, for any k, and therefore, this convergence holds in the C^∞-topology.

□

3.1 The one-parameter family $R_\lambda f$

To prove theorem 3.1, we need to consider the one-parameter family $R_\lambda f = f + \lambda$ (see [7, p.31]). We have the lemma:

Lemma 3.3. *Let $l \geq 3$ be an integer, $f \in D^l(\mathbb{T}^1)$, $0 < \eta \leq l-3$, $\alpha = \rho(f)$. Let $\tilde{\alpha}$ be an irrational number of constant type. There exists $\lambda_0 \in \mathbb{R}$ and a $C^{l-1-\eta}$-diffeomorphism h such that $h^{-1}R_{\lambda_0}fh = R_{\tilde{\alpha}}$. Moreover,*

$$\frac{|\lambda_0|}{\min Dh} \geq |\tilde{\alpha} - \alpha| \geq \frac{|\lambda_0|}{\|Dh\|_0}.$$

Proof. Let $\mu(\lambda) = \rho(R_\lambda f)$. μ is continuous, non-decreasing and $\mu(\mathbb{R}) = \mathbb{R}$ (see [7, p. 31]). Therefore, there exists $\lambda_0 \in \mathbb{R}$ such that $\tilde{\alpha} = \rho(R_{\lambda_0}f)$. Since $\tilde{\alpha}$ is of constant type, there exists a $C^{l-1-\eta}$-diffeomorphism h such that $h^{-1}R_{\lambda_0}fh = R_{\tilde{\alpha}}$ and that satisfies estimate (1) of theorem 2.3. By the mean value theorem, for any x, there is $c(x)$ such that:

$$\tilde{\alpha} + x - h^{-1}fh(x) = R_{\tilde{\alpha}}(x) - h^{-1}fh(x) = h^{-1}R_{\lambda_0}fh(x) - h^{-1}fh(x) = D(h^{-1})(c(x))\lambda_0.$$

By integrating this equation on an invariant measure of $h^{-1}fh$, we get lemma 3.3. Note that since $h \in D^1(\mathbb{T}^1)$, then $Dh(x) > 0$ for any x, and $\min Dh > 0$.

□

3.2 The speed of approximation of R_α

The proof of theorem 3.1 is also based on the lemma:

Lemma 3.4. *Let $l \geq 3$ be an integer, $f \in D^l(\mathbb{T}^1)$, $0 < \eta \leq l-3$, $\alpha = \rho(f)$. Let $\tilde{\alpha}$ be an irrational number of constant type, and let $\lambda_0 \in \mathbb{R}$ and h the $C^{l-1-\eta}$-diffeomorphism be given by lemma 3.3. Recall that C denotes a constant that only depends on u, $0 \leq u \leq l-2-\eta$. We have the estimate:*

$$\|h^{-1}fh - R_\alpha\|_u \leq C\|f\|_u^C \|h\|_{u+1}^C \frac{1}{(\min Dh)^C}|\tilde{\alpha} - \alpha|.$$

Before proving lemma 3.4, we show how theorem 3.1 is derived from it.

proof of theorem 3.1. If α is of constant type, then f is reducible and there is nothing to prove. Therefore, we can suppose that $V_\alpha(n) \to_{n\to+\infty} +\infty$. By applying theorem 2.3, there exists a real function $\tilde{F}$ strictly increasing with $V_\alpha(n)$, such that for α_n, and for its associated diffeomorphism h_n given by lemma 3.3, we have, for n sufficiently large:

$$\|h_n^{-1}fh_n - R_\alpha\|_{l-2-\eta} \leq \exp\left(\tilde{F}(V_\alpha(n))\right)|\alpha_n - \alpha|.$$

Let $F(n) = \tilde{F}^{-1}(n^{1/2})$. By extracting, we can suppose that $\lim \frac{V_\alpha(n)}{F(n)} = 0$. Therefore, $V_\alpha(n) \leq F(n)$ for n sufficiently large and therefore,

$$\tilde{F}(V_\alpha(n)) \leq n^{1/2}.$$

We get, for n sufficiently large,

$$\|h_n^{-1}fh_n - R_\alpha\|_{l-2-\eta} \leq e^{-\frac{n\log 2}{4}} \to_{n\to+\infty} 0.$$

Hence theorem 3.1.

□

Now, we show lemma 3.4:

proof of lemma 3.4. We need the Faa-di-Bruno formula (see e.g. [3]):

Lemma 3.5. *For every integer $u \geq 0$ and functions ϕ and ψ of class C^u, we have:*

$$D^u\left[\phi(\psi(x))\right] = \sum_{j=0}^{u} D^j\phi(\psi(x))B_{u,j}\left(D\psi(x), D^2\psi(x), \ldots, D^{(u-j+1)}\psi(x)\right).$$

The $B_{u,j}$ are the Bell polynomials, defined by $B_{u,0} = 1$ and, for $j \geq 1$:

$$B_{u,j}(x_1, x_2, \ldots, x_{u-j+1}) = \sum \frac{u!}{l_1!l_2!\cdots l_{u-j+1}!}\left(\frac{x_1}{1!}\right)^{l_1}\left(\frac{x_2}{2!}\right)^{l_2}\cdots\left(\frac{x_{u-j+1}}{(u-j+1)!}\right)^{l_{u-j+1}}.$$

The sum extends over all sequences $l_1, l_2, l_3, ..., l_{u-j+1}$ of non-negative integers such that:$l_1 + l_2 + ... = j$ and $l_1 + 2l_2 + 3l_3 + ... = u$.

Therefore, for any x, we have the estimate:

$$\left|B_{u,j}\left(D\psi(x), D^2\psi(x), \ldots, D^{(u-j+1)}\psi(x)\right)\right| \leq C\left(1 \vee \|\psi\|_u^j\right). \tag{2}$$

Combining this estimate with lemma 3.5, we obtain the corollary:

Corollary 3.6. *For every integer $u \geq 0$ and functions ϕ and ψ of class C^u, we have:*

$$\|\phi \circ \psi\|_u \leq C \max_{0\leq j\leq u} \|D^j\phi \circ \psi\|_0 \left(1 \vee \|\psi\|_u^u\right).$$

We apply this corollary to estimate $\|h^{-1}\|_u$. We let $\phi(x) = 1/x$ and $\psi = Dh \circ h^{-1}$. We observe that $D(h^{-1}) = \frac{1}{Dh\circ h^{-1}} = \phi \circ \psi$. Since there is x_0 such that $Dh(x_0) = 1$, then $\|Dh\|_0 \geq 1$ (and we also have $1 \geq \min Dh > 0$). Therefore, we get:

$$\|D(h^{-1})\|_u \leq C \max_{0\leq j\leq u} \frac{1}{\|\left(Dh \circ h^{-1}\right)^{j+1}\|_0}\|Dh \circ h^{-1}\|_u^C.$$

By corollary 3.6, we also have:

$$\|Dh \circ h^{-1}\|_u \leq C\|Dh\|_u\|h^{-1}\|_u^C.$$

By combining these two estimates, we get:

$$\|D(h^{-1})\|_u \leq C\frac{1}{(\min Dh)^C}\|Dh\|_u^C\|h^{-1}\|_u^C.$$

We iterate this estimate to estimate $\|h^{-1}\|_u$, for $u \geq 1$. We get:

$$\|h^{-1}\|_{u+1} \leq C\frac{1}{(\min Dh)^C}\|h\|_{u+1}^C\|h^{-1}\|_1^C. \tag{3}$$

Now, we estimate the C^u-distance of $h^{-1}fh$ to R_α. Let $\tilde{\alpha}, \lambda_0$ be as in lemma 3.3. We have:

$$h^{-1}fh - R_\alpha = h^{-1}fh - h^{-1}R_{\lambda_0}fh + R_{\tilde{\alpha}} - R_\alpha.$$

Therefore,

$$\|h^{-1}fh - R_\alpha\|_u \leq \|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u + |\tilde{\alpha} - \alpha|. \tag{4}$$

On the other hand, by the Faa-di-Bruno formula, we have:

$$D^u\left[h^{-1}fh - h^{-1}R_{\lambda_0}fh\right](x) = \sum_{j=0}^{u} B_{u,j}\left(D(fh)(x), ..., D^{u-j+1}(fh)(x)\right)$$
$$\left[D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x)+\lambda_0)\right].$$

Since $|D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x)+\lambda_0)| \leq \|D^{j+1}(h^{-1})\|_0|\lambda_0|$, then by applying estimate (2), we get:

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C\|f\circ h\|_u^C\|h^{-1}\|_{u+1}|\lambda_0|.$$

By applying corollary 3.6, we get:

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C\|f\|_u^C\|h\|_u^C\|h^{-1}\|_{u+1}|\lambda_0|.$$

By applying (3), we obtain:

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C\|f\|_u^C\|h\|_u^C\frac{1}{(\min Dh)^C}\|h\|_{u+1}^C\|h^{-1}\|_1^C|\tilde{\alpha}-\alpha|\|Dh\|_0$$

$$\|h^{-1}fh - h^{-1}R_{\lambda_0}fh\|_u \leq C\|f\|_u^C\|h\|_{u+1}^C\frac{|\tilde{\alpha}-\alpha|}{(\min Dh)^C}.$$

By estimate (4), we obtain:

$$\|h^{-1}fh - R_\alpha\|_u \leq C\|f\|_u^C\|h\|_{u+1}^C\frac{1}{(\min Dh)^C}|\tilde{\alpha}-\alpha|. \tag{5}$$

Hence lemma 3.4.

□

4 Application to commuting diffeomorphisms

Theorem 4.1. *There exists a numerical sequence $G(n)$, going to $+\infty$ as $n\to+\infty$, such that, for any $l\geq 3$ an integer, $f\in D^l(\mathbb{T}^1)$ of rotation number $\alpha\in\mathbb{R}$, $\eta>0$ and g of class C^l such that $fg=gf$, if*

$$\liminf\frac{V_\alpha(n)}{G(n)} = 0$$

then there exists two sequences of diffeomorphisms f_n and g_n that are $C^{l-1-\eta}$-conjugated to rotations, such that $f_ng_n = g_nf_n$, and with f_n and g_n converging respectively towards f and g in the $C^{l-2-\eta}$-norm.

Corollary 4.2. *There is a Baire-dense set $A_2\subset\mathbb{R}$ such that if $l\geq 3$ is an integer, $f\in D^l(\mathbb{T}^1)$ has a rotation number $\alpha\in A_2$, g is of class C^l such that $fg=gf$ and $\eta\in\mathbb{R}_+$, then there exists two sequences of diffeomorphisms f_n and g_n that are $C^{l-1-\eta}$-conjugated to rotations, such that $f_ng_n=g_nf_n$ and with f_n and g_n converging respectively towards f and g in the $C^{l-2-\eta}$-norm.*

We derive theorem 1.2 from corollary 4.2 by following the same argument as in the proof of theorem 1.1.

4.1 The speed of approximation of g by a linearizable and commuting diffeomorphism

To prove theorem 4.1, we consider $(h_n)_{n\geq 0}$, the sequence of conjugating diffeomorphisms constructed in the proof of theorem 3.1, $(\lambda_n)_{n\geq 0}$ the associated sequence of real numbers such that $f_n = R_{\lambda_n} f = h_n R_{\alpha_n} h_n^{-1}$. We also consider $g_n' = h_n^{-1} g h_n$ and $g_n = h_n R_{g_n'(0)} h_n^{-1}$. The diffeomorphisms f_n and g_n commute, and $f_n \to f$ in the $C^{l-2-\eta}$-norm. To prove theorem 4.1, it suffices to show that $g_n \to g$ in the $C^{l-2-\eta}$-norm. This convergence is based on the lemma:

Lemma 4.3. *Let $l \geq 3$ be an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in \mathbb{R}$, $\eta > 0$, $0 \leq u \leq l-2-\eta$, and $g \in D^l(\mathbb{T}^1)$ be such that $fg = gf$. Let $(q_t)_{t\geq 0}$ be the sequence of denominators of the convergents of α, and let $r \geq 0$ be an integer. Let $\tilde{\alpha}$ be an irrational number of constant type, $\lambda_0 \in \mathbb{R}$ the associated number and h the associated $C^{l-1-\eta}$ diffeomorphism given by lemma 3.3. Let $f' = h^{-1} f h$ and $g' = h^{-1} g h$. We have the estimate:*

$$\|g - hR_{g'(0)}h^{-1}\|_u \leq C\|h\|_{u+1}^C \|f\|_u^C \|g\|_{u+1}^C \left(\frac{1}{q_r} + |\tilde{\alpha} - \alpha| \left(\frac{(C\|h\|_{u+1}\|f\|_{u+1})^{Cq_r}}{(\min Dh)^C}\right)\right).$$

To show this lemma, the basic idea is the following: we approach modulo 1 points $x \in \mathbb{R}$ by $p(x)\alpha$ mod 1, where $p(x) \leq q_r$ is an integer, and where the integer r will be fixed later. We have a control of $|x - p(x)\alpha|$ mod 1 in function of q_r. Then, by using the assumption of commutation $g' f'^p = f'^p g'$, we can write:

$$g'(x) - R_{g'(0)}(x) = g'(x) - g'(p\alpha) + g'(p\alpha) - g' f'^p(0) + f'^p g'(0) - R_{p\alpha}(g'(0)) + R_{g'(0)}(p\alpha) - R_{g'(0)}(x).$$

We use the distance of f'^p to $R_{p\alpha}$, which depends on q_r and the norm of $f' - R_\alpha$. This distance has been estimated in the proof of the result of quasi-reductibility. We also use C^k analogues, $k \geq 2$, of the mean value theorem, obtained with the Faa-di-Bruno formula. This allows to estimate the norm of $g - hR_{g'(0)}h^{-1}$ in function of the norm of $g' - R_{g'(0)}$.

To obtain theorem 4.1 from lemma 4.3, we take $\tilde{\alpha} = \alpha_n$, and we consider the associated sequences $f_n, g_n, f_n', g_n', h_n$. The integer q_r must be chosen sufficiently large with respect to the conjugacy h_n, so that $|x - p\alpha|$ mod 1 is sufficiently small. However, this integer q_r must not be too large, to keep the norm of $f_n'^p - R_{p\alpha}$ sufficiently small. This integer q_r is controlled with $\sup_{k\leq r} a_k$, which itself controls the norm of h_r. Thus, it suffices to properly choose the integer r in function of n, in order to obtain the convergence of g_n towards g.

Proof of theorem 4.1. Assuming lemma 4.3, we show theorem 4.1.

Let $\tilde{\alpha} = \alpha_n$ and h_n be the associated diffeomorphism given by lemma 3.3. Since $V_\alpha(n) \to +\infty$, by applying the estimate for the conjugacy h_n, there exists $\tilde{G}(x)$ strictly increasing with x such that, for n sufficiently large:

$$\|g - h_n R_{g_n'(0)} h_n^{-1}\|_{l-2-\eta} \leq e^{C\tilde{G}(V_\alpha(n))} \left(\frac{1}{q_r} + \frac{e^{C\tilde{G}(V_\alpha(n))q_r}}{2^n}\right).$$

Moreover, since $q_n = a_n q_{n-1} + q_{n-2}$, and $q_{n-2} \leq q_{n-1}$, then

$$(\sqrt{2})^{n-1} \leq q_n \leq \prod_{k=1}^{n} (a_k + 1). \tag{6}$$

Therefore, we get:

$$\|g - h_n R_{g'_n(0)} h_n^{-1}\|_{l-2-\eta} \leq e^{C\tilde{G}(V_\alpha(n)) - \frac{1}{2}(r-1)\log 2} + e^{C\tilde{G}(V_\alpha(n)) + C\tilde{G}(V_\alpha(n))(V_\alpha(r)+1)^r - n\log 2}. \quad (7)$$

Let $G(n) = \tilde{G}^{-1}((\log n)^{1/2})$. By extracting in the sequence $V_\alpha(n)/G(n)$, we can suppose that:

$$\frac{V_\alpha(n)}{G(n)} \to 0.$$

Therefore, for n sufficiently large, we have:

$$\tilde{G}(V_\alpha(n)) \leq (\log n)^{1/2}.$$

Moreover, for n sufficiently large, we can take an integer r_n such that:

$$(\log n)^{3/4} \leq r_n \leq (\log n)^{7/8}.$$

We get:

$$(V_\alpha(r_n) + 1)^{r_n} = e^{r_n \log(V_\alpha(r_n)+1)} \leq e^{(\log n)^{15/16}}.$$

The first term in estimate (7) tends towards 0. Moreover, since,for n sufficiently large,

$$(\log n)^{1/2} e^{(\log n)^{15/16}} \leq \frac{n}{2} \log 2$$

then the second term also tends towards 0. Hence theorem 4.1.

□

4.2 Higher-order analogous of the mean value theorem

Proof of lemma 4.3. We need two higher-order analogous of the mean value theorem. The first one is:

Lemma 4.4. *Let $u \geq 0$, $s, t \in D^u(\mathbb{T}^1)$. Let $\delta \in \mathbb{R}$. We have:*

$$\|st - R_\delta t\|_u \leq C\|s\|_{u+1}\|s - R_\delta\|_u \|t\|_u^u.$$

Observe the presence of the term $\|s\|_{u+1}$, which is absent in the mean value formula. This is because of the estimate (2) on the Bell polynomial, in the Faa-di-Bruno formula.

Proof. If $u = 0$, the estimate is trivial. We suppose $u \geq 1$. For any $x \in \mathbb{R}$, the Faa-di-Bruno formula gives:

$$D^u(st)(x) - D^u(R_\delta t)(x) = \sum_{j=0}^{u} \left((D^j s)(t(x)) - (D^j R_\delta)(t(x))\right) B_{u,j}\left(Dt(x), ..., D^{u-j+1}t(x)\right).$$

Therefore, by estimate (2), and since $\|t\|_u \geq 1$,

$$|D^u(st)(x) - D^u(R_\delta t)(x)| \leq C\|s\|_{u+1}\|s - R_\delta\|_u \|t\|_u^u.$$

Hence lemma 4.4.

□

The second higher-order analogous of the mean value theorem is:

Lemma 4.5. *Let $u \geq 0$, $s \in D^{u+1}(\mathbb{T}^1)$, $t \in D^u(\mathbb{T}^1)$, $\delta \in \mathbb{R}$. We have:*

$$||st - sR_\delta||_u \leq C||s||_{u+1}||t||_u^u||t - R_\delta||_u.$$

Observe the presence of the term $||t||_u$, which is absent in the mean value formula. As in lemma 4.4, this is because of an estimate on the Bell polynomial, in the Faa-di-Bruno formula.

Proof. If $u = 0$, the estimate holds. We suppose $u \geq 1$. We use the following lemma:

Lemma 4.6. *Let $u \geq 1$, $j \leq u$ be integers and $a_1, ..., a_{u-j+1}, x_1, ..., x_{u-j+1} \geq 0$. Let $x \geq \max\{|x_k| \vee 1;\ 1 \leq k \leq u-j+1\}$ and let $a \geq \max\{|a_k|;\ 1 \leq k \leq u-j+1\}$. Let $B_{u,j}$ be a Bell polynomial. We have:*

$$|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq Ca(x+a)^u.$$

Proof. Let $p \geq 1$ and $l_1, ..., l_p$ be integers. Then we have:

$$(x_1+a_1)^{l_1}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_{i-1}}(x_i+a_i)^{l_i}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_i^{l_i}(x_{i+1}+a_{i+1})^{l_{i+1}}...(x_p+a_p)^{l_p}$$

$$(x_1+a_1)^{l_1}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_{i-1}}(x_{i+1}+a_{i+1})^{l_{i+1}}...(x_p+a_p)^{l_p}\left[(x_i+a_i)^{l_i} - x_i^{l_i}\right]$$

(with the conventions $x_1^{l_1}...x_0^{l_0} = 1$ and $x_{p+1}^{l_{p+1}}...x_p^{l_p} = 1$).

Since $(x_i + a_i)^{l_i} - x_i^{l_i} \leq l_i|a_i|(|x_i| + |a_i|)^{l_i-1} \leq l_i a(|x_i| + a)^{l_i-1}$, $1 \leq l_i \leq u$ and $x + a \geq 1$ (because $x \geq 1$), we obtain:

$$|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq a(u-j+1)uB_{u,j}(x+a, ..., x+a).$$

By the formula giving the Bell polynomials, we have:

$$B_{u,j}(x + a, ..., x + a) \leq C(x + a)^u.$$

□

To show lemma 4.5, For any $0 \leq v \leq u$, we write:

$$D^v(st)(x) - D^v(sR_\delta)(x) = \sum_{j=0}^{v} D^j s(t(x))\left[B_{v,j}\left(Dt(x), ..., D^{v-j+1}t(x)\right) - B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)\right] +$$

$$\left[D^j s(t(x)) - D^j s(R_\delta(x))\right] B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right).$$

We apply lemma 4.6 with $a = ||t - R_\delta||_u$ and $x = ||R_\delta||_u \geq 1$. Since $t \in D^u(\mathbb{T}^1)$, then $||t||_u \geq 1$. We get:

$$\left|B_{v,j}\left(Dt(x), ..., D^{v-j+1}t(x)\right) - B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)\right| \leq C||t - R_\delta||_u(1 + ||t - R_\delta||_u)^u$$

$$\left|B_{v,j}\left(Dt(x), ..., D^{v-j+1}t(x)\right) - B_{v,j}\left(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)\right)\right| \leq C||t - R_\delta||_u(2 + ||t||_u)^u \leq C||t - R_\delta||_u||t||_u^u.$$

□

4.3 Successive estimates

To prove lemma 4.3, we also need these successive estimates:

Lemma 4.7. *Let $l \geq 3$ be an integer, $f \in D^l(\mathbb{T}^1)$ of rotation number $\alpha \in \mathbb{R}$, $\eta > 0$, $0 \leq u \leq l-2-\eta$, and $g \in D^l(\mathbb{T}^1)$ be such that $fg = gf$. Let $(q_t)_{t\geq 0}$ be the sequence of denominators of the convergents of α. Let $\tilde{\alpha}$ be an irrational number of constant type, $\lambda_0 \in \mathbb{R}$ the associated number and h the associated $C^{l-1-\eta}$ diffeomorphism given by lemma 3.3. Let $f' = h^{-1}fh$ and $g' = h^{-1}gh$. We have the estimates:*

$$A_{1,u} = \|h^{-1}\|_u \leq C\|h\|_u^C \frac{1}{(\min Dh)^C} \tag{8}$$

$$A_{2,u} = \|f'\|_u \leq CA_{1,u}\|f\|_u^C\|h\|_u^C \tag{9}$$

$$A_{3,u}(m) = \|f'^m\|_u \leq C^m A_{2,u}^{mC} \tag{10}$$

$$A_{4,u} = \|f' - R_\alpha\|_u \leq C\|h\|_{u+1}^C\|f\|_u^C \frac{1}{(\min Dh)^C}|\tilde{\alpha} - \alpha| \tag{11}$$

$$A_{5,u}(m) = \|f'^m - R_{m\alpha}\|_u \leq mCA_{4,u}A_{2,u}^C \max_{k\leq m-1} A_{3,u+1}(k) \tag{12}$$

$$A_{6,u} = \|g'\|_u \leq CA_{1,u}\|g\|_u^C\|h\|_u^C \tag{13}$$

and for any integer $r \geq 0$, we have:

$$A_{7,u} = \|g' - R_{g'(0)}\|_u \leq \frac{A_{6,u+1}+1}{q_r} + \max_{m\leq 2q_r}\left(A_{6,u+1}A_{3,u}^C(m)A_{5,u}(m) + A_{6,u}^C A_{3,u+1}(m)A_{5,u}(m)\right) \tag{14}$$

$$A_{8,u} = \|g'h^{-1} - R_\alpha h^{-1}\|_u \leq CA_{6,u+1}A_{7,u}A_{1,u}^C \tag{15}$$

$$A_{9,u} = \|hg'h^{-1} - hR_{g'(0)}h^{-1}\|_u \leq C\|g\|_u^C A_{8,u}A_{1,u}^C\|h\|_{u+1}. \tag{16}$$

The crucial estimate is (14), which is obtained by approaching modulo 1 each $x \in \mathbb{R}$ by a $m(x)\alpha$, with $m(x) \leq q_r$. If q_r increases, $x - m(x)\alpha$ is smaller modulo 1, but the bound on $A_{3,u}(m(x))$ and $A_{5,u}(m(x))$ increases. In the proof of theorem 4.1, we make a proper choice of r (and q_r).

estimate (11) corresponds to estimate (5) of the proof of the result of quasi-reducibility.

The other estimates, namely, estimates (8),(9),(10), (12),(13), (15) and (16) are derived from applications of the Faa-di-Bruno formula: either corollary 3.6, lemma 4.4 or lemma 4.5.

Proof of lemma 4.7. For $A_{1,u}$, by estimate (3), we have:

$$\|h^{-1}\|_u \leq C\|h\|_u^C \frac{1}{(\min Dh)^C}.$$

Hence estimate (8).

For $A_{2,u}$, by applying corollary 3.6 twice, we have,

$$\|f'\|_u \leq CA_{1,u}\|f\|_u^C\|h\|_u^C.$$

Hence estimate (9).

For $A_{3,u}$, by applying corollary 3.6 again, we have, for any m,

$$\|f'^{m+1}\|_u \leq C\|f'^m\|_u\|f'\|_u^C$$

and therefore, by iteration, we get:

$$\|f'^m\|_u \leq C^m\|f'\|_u^{mC}.$$

Hence (10).

estimate (11) is a direct application of estimate (5).

For estimate (12), we observe that for any $0 \leq v \leq u$:

$$D^v f'^m - D^v R_{m\alpha} = D^v \sum_{k=0}^{m-1} f'^{m-k} R_{k\alpha} - f'^{m-k-1} R_{(k+1)\alpha}$$

$$D^v f'^m - D^v R_{m\alpha} = \sum_{k=0}^{m-1} D^v \left(f'^{m-k-1} f'\right) R_{k\alpha} - D^v \left(f'^{m-k-1} R_\alpha\right) R_{k\alpha}.$$

By applying lemma 4.5, and by noting that for any k, $\|f'^{m-k-1}\|_{u+1} \leq \max_{0\leq k\leq m-1} \|f'^k\|_{u+1}$, we get:

$$\|f'^m - R_{m\alpha}\|_u \leq mC\|f'\|_u^C \max_{0\leq k\leq m-1} \|f'^k\|_{u+1}\|f' - R_\alpha\|_u.$$

Hence (12).

For $A_{6,u}$, estimate (13) is the same as (9):

$$\|g'\|_u \leq C\|h^{-1}\|_u\|g\|_u^C\|h\|_u^C.$$

Hence (13).

For $A_{7,u}$, let $m \geq 0$ and $u \geq v \geq 1$. For any x, $D^v R_\alpha(x) = \int_0^1 D^v g'(y)\, dy$. Therefore,

$$|D^v g'(x) - D^v R_\alpha(x)| = \left|D^v g'(x) - \int_0^1 D^v g'(y)\, dy\right| =$$
$$\left|\int_0^1 \left(D^v g'(x) - D^v g'(y)\right) dy\right| \leq \max_{x,y\in[0,1]} |D^v g'(x) - D^v g'(y)|.$$

On the other hand, we have:

$$D^v g'(x) - D^v g'(y) = D^v g'(x) - D^v g'(y + m\alpha) + D^v g'(R_{m\alpha}(y)) - D^v(g' f'^m(y)) + D^v(f'^m g'(y)) - D^v g'(y).$$

Moreover, we have:

$$|D^v g'(x) - D^v g'(y + m\alpha)| \leq |D^{u+1} g'|_0 |x - y - m\alpha|.$$

By lemma 4.5, we also have:

$$|D^v g'(R_{m\alpha}(y)) - D^v(g' f'^m(y))| \leq C\|g'\|_{u+1}\|f'^m\|_u^C \|f'^m - R_{m\alpha}\|_u.$$

Finally, by lemma 4.4, we have:

$$|D^v(f'^m g'(y)) - D^v(R_{m\alpha} g'(y))| \leq C\|f'^m\|_{u+1}\|f'^m - R_{m\alpha}\|_u \|g'\|_u^C.$$

Since $R_{m\alpha} g'(y) = g'(y) + m\alpha$, and $v \geq 1$, then $D^v(R_{m\alpha} g'(y)) = D^v(R_{m\alpha} g'(y))$. Therefore, the same estimate holds for $|D^v(f'^m g'(y)) - D^v(g'(y))|$.

By combining these estimates, we obtain:

$$|D^v g'(x) - D^v g'(y)| \leq \|g'\|_{u+1}|x - y - m\alpha| + C\|g'\|_{u+1}\|f'^m\|_u^C\|f'^m - R_{m\alpha}\|_u + C\|f'^m\|_{u+1}\|f'^m - R_{m\alpha}\|_u\|g'\|_u^C.$$

Moreover, for any $r \geq 0$, any $x, y \in \mathbb{R}$, there is an integer $m(x, y) \leq 2q_r$, there are real numbers x', y' such that $x' - x \in \mathbb{Z}$, $y' - y \in \mathbb{Z}$ and such that $|x' - y' - m(x, y)\alpha| \leq 1/q_r$. Since $v \geq 1$, then $|D^v g'(x) - D^v g'(y)| = |D^v g'(x') - D^v g'(y')|$. We apply the former estimate with x' and y' and we get:

$$\max_{1 \leq v \leq u} \left\|D^v g' - D^v R_{g'(0)}\right\|_0 \leq \frac{A_{6,u+1} + 1}{q_r} + \max_{m \leq 2q_r} \left(A_{6,u+1} A_{3,u}^C(m) A_{5,u}(m) + A_{6,u}^C A_{3,u+1}(m) A_{5,u}(m)\right).$$

If $v = 0$, we note that for any $r \geq 0$, any $x \in \mathbb{R}$, there is an integer $m(x) \leq q_r$ and a real number $x' \in \mathbb{R}$ such that $x' - x \in \mathbb{Z}$, and such that $|x' - m(x)\alpha| \leq 1/q_r$. Moreover, we have: $g'(x) - R_{g'(0)}(x) = g'(x') - R_{g'(0)}(x')$, and

$$g'(x') - R_{g'(0)}(x') = g'(x') - g'(m\alpha) + g'(m\alpha) - g' f'^m(0) + f'^m g'(0) - R_{m\alpha}(g'(0)) + R_{g'(0)}(m\alpha) - R_{g'(0)}(x').$$

Hence estimate (14).

For $A_{8,u}$, estimate (15) follows immediately from lemma 4.4.

For $A_{9,u}$, let $x \in \mathbb{R}$. Let $0 \leq v \leq u$. By the Faa-di-Bruno formula:

$$D^v\left(hg'h^{-1}\right)(x) - D^v\left(hR_{g'(0)}h^{-1}\right)(x) = \sum_{j=0}^{v} D^j h(g'h^{-1}(x)) B_{v,j}\left(D\left(g'h^{-1}\right)(x), ..., D^{v-j+1}\left(g'h^{-1}(x)\right)\right) - D^j h(g'h^{-1}(x)) B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x), ..., D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right)$$

$$= \sum_{j=0}^{v} D^j h(g'h^{-1}(x))$$
$$\left[B_{v,j}\left(D\left(g'h^{-1}\right)(x),...,D^{v-j+1}\left(g'h^{-1}(x)\right)\right) - B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x),...,D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right)\right] -$$
$$\left[D^j h(R_{g'(0)}h^{-1}(x)) - D^j h(g'h^{-1}(x))\right] B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x),...,D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right).$$

Since $\|h^{-1}\|_u \geq 1$, then lemma 4.6 gives,

$$\left|B_{v,j}\left(D\left(g'h^{-1}\right)(x),...,D^{v-j+1}\left(g'h^{-1}(x)\right)\right) - B_{v,j}\left(D\left(R_{g'(0)}h^{-1}\right)(x),...,D^{v-j+1}\left(R_{g'(0)}h^{-1}(x)\right)\right)\right| \leq$$
$$C\|g'h^{-1}\|_u^C \|g'h^{-1} - R_{g'(0)}h^{-1}\|_u.$$

Since $g'h^{-1} = h^{-1}g$ and $\|h^{-1}g\|_u \leq C\|h^{-1}\|_u\|g\|_u^C$, we get,

$$\left|D^v\left(hg'h^{-1}\right)(x) - D^v\left(hR_{g'(0)}h^{-1}\right)(x)\right| \leq$$
$$C\|g\|_u^C\|h\|_u\|h^{-1}\|_u^C\|g'h^{-1} - R_{g'(0)}h^{-1}\|_u + C\|h\|_{u+1}\|g'h^{-1} - R_{g'(0)}h^{-1}\|_u\|h^{-1}\|_u^C.$$

Hence estimate (16). This completes the proof of lemma 4.7.

□

By combining these estimates, we obtain:

$$A_{9,u} \leq CA_{1,u+1}^C\|h\|_{u+1}^C\|g\|_{u+1}^C\left(\frac{1}{q_r} + \max_{m\leq 2q_r}\left(A_{3,u+1}^C(m)A_{5,u}(m)\right)\right)$$

$$A_{9,u} \leq C\|h\|_{u+1}^C\|f\|_u^C\|g\|_{u+1}^C\left(\frac{1}{q_r} + |\tilde{\alpha} - \alpha|\left(\frac{(C\|h\|_{u+1}\|f\|_{u+1})^{Cq_r}}{(\min Dh)^C}\right)\right).$$

Hence lemma 4.3. Notice the loss of one derivative for h.

□

A Appendix: proof of the C^∞-genericity of $(O^1_{\alpha,\beta})^c$ in $F_{\alpha,\beta}$

To show that $(O^1_{\alpha,\beta})^c$ is C^∞-generic in $F_{\alpha,\beta}$, we slightly modify [7, p.160, p.167]. Let $H : F_{\alpha,\beta} \to \mathbb{R}_+ \cup \{+\infty\}$ be defined by $H(f,g) = \sup_{n\geq 1}(\|Df^n\|_0, \|Dg^n\|_0)$.

The map H is lower semi-continuous, because it is an upper bound of a family of continuous maps. Therefore, $\{(f,g) \in F_{\alpha,\beta} | H(f,g) > n\}$ is open, and

$H^{-1}(+\infty) = \cap_{n\geq 1}\{(f,g) \in F_{\alpha,\beta} | H(f,g) > n\}$ is a G_δ-set (i.e. a countable intersection of open sets).

By [7, p.52], $(O^1_{\alpha,\beta})^c = H^{-1}(+\infty)$ (f and g are not C^1-conjugated to a rotation if and only if $H(f,g) = +\infty$). By the first part of remark 1.4, $S_{\alpha,\beta} \subset (O^1_{\alpha,\beta})^c$ is C^∞-dense. Since C^1-open sets are C^∞-open (if ϕ_n does not converge to ϕ in the C^1 norm, then ϕ_n does not converge to ϕ in the C^∞ norm), we conclude that $(O^1_{\alpha,\beta})^c$ is C^∞-generic in $F_{\alpha,\beta}$.

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